(T1) **True or False**
Determine if each of the following statements is True or False. In the Summary Answersheet, tick the correct answer (TRUE / FALSE) for each statement. No justification is necessary for this question.

(T1.1) In a photograph of the clear sky on a Full Moon night, with a sufficiently long exposure, the colour of the sky would appear blue as in daytime. **2**

(T1.2) An astronomer at Bhubaneswar marks the position of the Sun on the sky at 05: 00 UT every day of the year. If the Earth's axis were perpendicular to its orbital plane, these positions would trace an arc of a great circle. **2**

(T1.3) If the orbital period of a certain minor body around the Sun, in the ecliptic plane, is less than the orbital period of Uranus, then its orbit must necessarily be fully inside the orbit of Uranus. **2**

(T1.4) The centre of mass of the solar system is inside the Sun at all times. **2**

(T1.5) A photon is moving in free space. As the Universe expands its momentum decreases. **2**

(T2) **Gases on Titan** **10**
Gas particles in a planetary atmosphere have a wide distribution of speeds. If the r.m.s. (root mean square) thermal speed of particles of a particular gas exceeds $1/6$ of the escape speed, then most of that gas will escape from the planet. What is the minimum atomic weight (relative atomic mass), A_{min} of an ideal monatomic gas, so that it remains in the atmosphere of Titan?

Given the mass of Titan $M_{\mathrm{T}} = 1.23\times10^{23}$ kg, the radius of Titan $R_{\mathrm{T}} = 2575$ km, the surface temperature of Titan $T_{\mathrm{T}} = 93.7$ K.

(T3) **Early Universe**
Cosmological models indicate that the radiation energy density, ρ_{r}, in the Universe is proportional to $(1+z)^4$ and the matter energy density, ρ_{m}, is proportional to $(1+z)^3$, where z is the redshift. The dimensionless density parameter, Ω, is given by $\Omega = \rho/\rho_{\mathrm{c}}$, where ρ_{c} is the critical energy density of the Universe. In the present Universe, the density parameters corresponding to radiation and matter are $\Omega_{\mathrm{r}_0} = 10^{-4}$ and $\Omega_{\mathrm{m}_0} = 0.3$, respectively.

(T3.1) Calculate the redshift, z_{e}, at which radiation and matter energy densities were equal. **3**

(T3.2) Assuming that the radiation from the early Universe has a blackbody spectrum with a temperature of 2.732 K, estimate the temperature, T_{e}, of the radiation at redshift z_{e}. **4**

(T3.3) Estimate the typical photon energy, E_{ν} (in eV), of the radiation emitted at redshift z_{e}. **3**

(T4) **Shadows** **10**
An observer in the northern hemisphere noticed that the length of the shortest shadow of a 1.000 m vertical stick on a day was 1.732 m. On the same day, the length of the longest shadow of the same vertical stick was measured to be 5.671 m.

Find the latitude, ϕ, of the observer and declination of the Sun, $\delta_{\odot}$, on that day. Assume the Sun to be a point source and ignore atmospheric refraction.

(T5) **GMRT beam transit** **10**
The Giant Metrewave Radio Telescope (GMRT), one of the world's largest radio telescopes at metre wavelengths, is located in western India (latitude: $19°6'$ N, longitude: $74°3'$ E). The GMRT consists of 30 dish antennas, each with a diameter of 45.0 m. A single dish of GMRT was held fixed with its axis pointing at a zenith angle of $39°42'$ on the northern meridian, such that a radio point source would pass across the diameter of the beam when transiting the meridian.

What is the duration T_{transit} for which this source is within the FWHM (full width at half maximum) of the beam of a single GMRT dish , when observing at 200 MHz?

Hint: The FWHM size of the beam of a radio dish operating at a given frequency corresponds to the angular resolution of the dish. Assume uniform illumination.

(T6) **Cepheid Pulsations**

The star β-Doradus is a Cepheid variable star with a pulsation period of 9.84 days. The simplifying assumption is made that the star is brightest when it is most contracted (radius R_1) and it is faintest when it is most expanded (radius R_2). For simplicity, assume that the star maintains its spherical shape and behaves as a perfect black body at every instant during the entire cycle. The bolometric magnitude of the star varies from 3.46 to 4.08. From Doppler measurements, we know that during pulsation the stellar surface expands or contracts at an average radial speed of 12.8 km s^{-1}. Over the period of pulsation, the peak of thermal radiation (intrinsic) of the star varies from 531.0 nm to 649.1 nm.

(T6.1) Find the ratio of radii of the star in its most contracted and most expanded states (R_1/R_2). **7**

(T6.2) Find the radii of the star (in metres) in its most contracted and most expanded states (R_1 and R_2). **3**

(T6.3) Calculate the flux of the star, F_2, when it is in its most expanded state. **5**

(T6.4) Find the distance to the star, D_{star}, in parsecs. **5**

(T7) **Telescope optics**

In a particular ideal refracting telescope of focal ratio $f/5$, the focal length of the objective lens is 100 cm and that of the eyepiece is 1 cm.

(T7.1) What is the angular magnification, m_0, of the telescope? What is the length of the telescope, L_0, i.e. the distance between its objective and eyepiece? **4**

An introduction of a concave lens (Barlow lens) between the objective lens and the prime focus is a common way to increase the magnification without a large increase in the length of the telescope. A Barlow lens of focal length 1 cm is now introduced between the objective and the eyepiece to double the magnification.

(T7.2) At what distance, d_{B}, from the prime focus must the Barlow lens be kept in order to obtain this desired double magnification? **6**

(T7.3) What is the increase, ΔL, in the length of the telescope? **4**

A telescope is now constructed with the same objective lens and a CCD detector placed at the prime focus (without any Barlow lens or eyepiece). The size of each pixel of the CCD detector is 10 μm.

(T7.4) What will be the distance in pixels between the centroids of the images of the two stars , n_{p}, on the CCD, if they are $20''$ apart on the sky? **6**

(T8) **U-Band photometry**

A star has an apparent magnitude $m_{\text{U}} = 15.0$ in the U-band. The U-band filter is ideal, i.e., it has perfect (100%) transmission within the band and is completely opaque (0% transmission) outside the band. The filter is centered at 360 nm, and has a width of 80 nm. It is assumed that the star also has a flat energy spectrum with respect to frequency. The conversion between magnitude, m, in any band and flux density, f, of a star in Jansky (1 Jy $= 1\times10^{-26}$ W Hz^{-1} m^{-2}) is given by

$$f = 3631\times10^{-0.4m}\ \text{Jy}$$

(T8.1) Approximately how many U-band photons, N_0, from this star will be incident normally on a 1 m^2 area at the top of the Earth's atmosphere every second? **8**

This star is observed in the U-band using a ground based telescope, whose primary mirror has a diameter of 2.0 m. Atmospheric extinction in the U-band during the observation is 50%. You may assume that the seeing is diffraction limited. The average surface brightness of night sky in the U-band was measured to be 22.0 mag/arcsec^2.

(T8.2) What is the ratio, R, of number of photons received per second from the star to that received from the sky, when measured by a circular aperture of diameter $2''$? **8**

(T8.3) In practice, only 20% of U-band photons falling on the primary mirror are detected. How many photons, N_{t}, from the star are detected per second? **4**

(T9) **Mars Orbiter Mission**

India's Mars Orbiter Mission (MOM) was launched using the Polar Satellite Launch Vehicle (PSLV) on 5 November 2013. The dry mass of MOM (body + instruments) was 500 kg and it carried fuel of mass 852 kg. It was initially placed in an elliptical orbit around the Earth with perigee at a height of 264.1 km and apogee at a height of 23903.6 km, above the surface of the Earth. After raising the orbit six times, MOM was transferred to a trans-Mars injection orbit (Hohmann orbit).

The first such orbit-raising was performed by firing the engines for a very short time near the perigee. The engines were fired to change the orbit without changing the plane of the orbit and without changing its perigee. This gave a net impulse of 1.73×10^{5} kg m s^{-1} to the satellite. Ignore the change in mass due to the burning of fuel.

(T9.1) What is the height of the new apogee, h_{a} above the surface of the Earth, after this engine burn? **14**

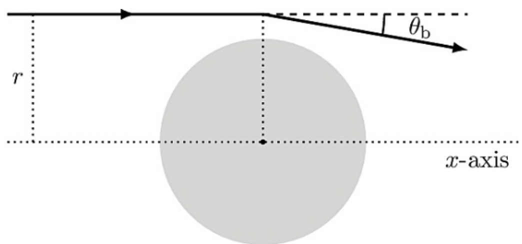
(T9.2) Find the eccentricity (e) of the new orbit after the burn and the new orbital period (P) of MOM in hours. **6**

(T10) **Gravitational Lensing Telescope**

Einstein's General Theory of Relativity predicts the bending of light around massive bodies. For simplicity, we assume that the bending of light happens at a single point for each light ray, as shown in the figure. The angle of bending, θ_{b}, is given by

$$\theta_{\mathrm{b}} = \frac{2R_{\mathrm{sch}}}{r}$$

where R_{sch} is the Schwarzschild radius associated with the gravitational body. The distance of the incoming light ray from the parallel x-axis passing through the centre of the body is the "impact parameter", r.



A massive body thus behaves somewhat like a focusing lens. The light rays coming from infinite distance beyond a massive body, and having the same impact parameter r, converge at a point along the axis, at a distance f_r from the centre of the massive body. An observer at that point will benefit from huge amplification due to this gravitational focusing. The massive body in this case is being used as a Gravitational Lensing Telescope for the amplification of distant signals.

(T10.1) Consider our Sun acting as a gravitational lensing telescope. Calculate the shortest distance, f_{min}, from the centre of the Sun (in A. U.) at which the light rays can be focused. **6**

(T10.2) Consider a small circular detector of radius a, at a distance f_{min}, centered on the x-axis and perpendicular to it. Only the light rays which pass within a certain annulus (ring) of width h (where $h \ll R_{\odot}$) around the Sun would encounter the detector. The amplification factor at the detector is defined as the ratio of the intensity of the light incident on it, in the presence of the Sun, and the intensity in the absence of the Sun. **8**

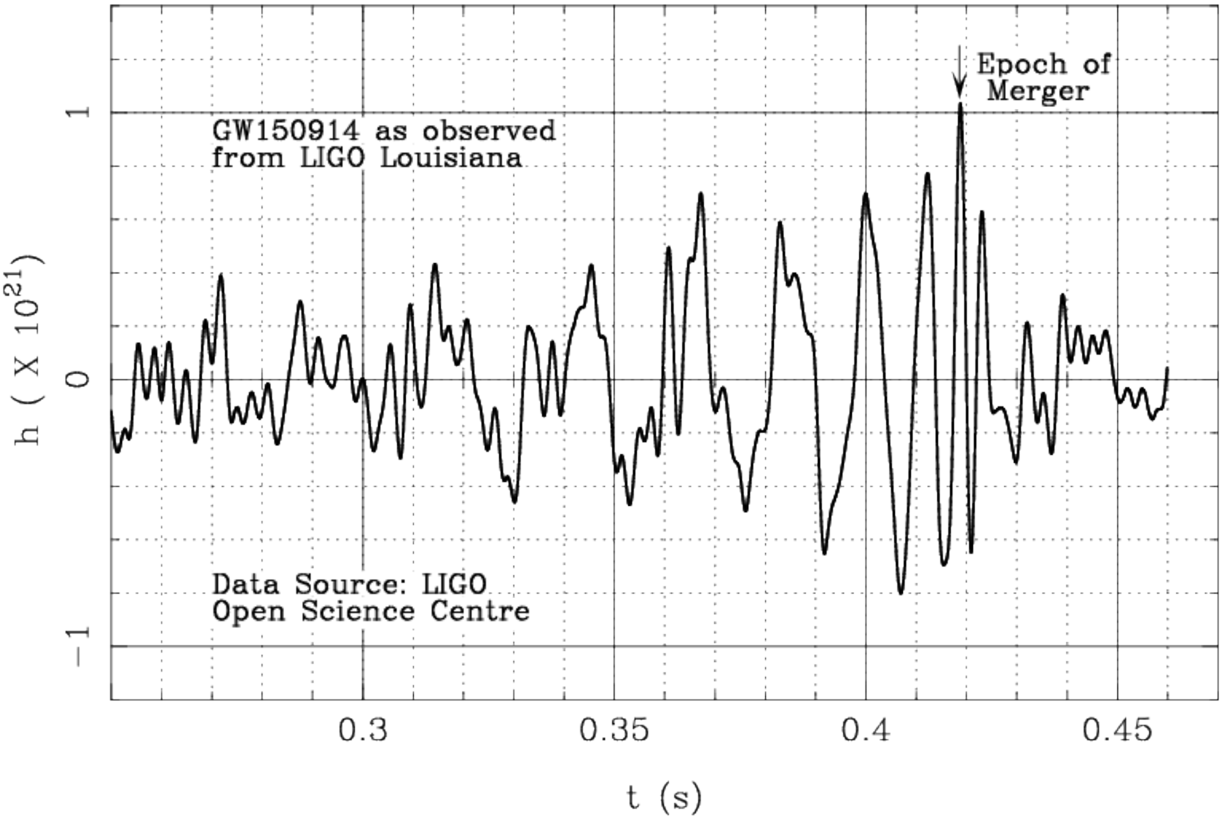
Express the amplification factor, A_{m}, at the detector in terms of $R_{\odot}$ and a.

(T10.3) Consider a spherical mass distribution, such as dark matter in a galaxy cluster, through which light rays can pass while undergoing gravitational bending. Assume for simplicity that for the gravitational bending with impact parameter, r, only the mass $M(r)$ enclosed inside the radius r is relevant. **6**

What should be the mass distribution, $M(r)$, such that the gravitational lens behaves like an ideal optical convex lens?

(T11) **Gravitational Waves**

The first signal of gravitational waves was observed by two advanced LIGO detectors at Hanford and Livingston, USA, in September 2015. One of these measurements (strain vs time in seconds) is shown in the accompanying figure. In this problem, we will interpret this signal in terms of a small test mass m orbiting around a large mass M (i.e., $m \ll M$), by considering several models for the nature of the central mass.



The test mass loses energy due to the emission of gravitational waves. As a result, the orbit keeps on shrinking, until the test mass reaches the surface of the object, or in the case of a black hole, the innermost stable circular orbit – ISCO – which is given by $R_{\text{ISCO}} = 3R_{\text{sch}}$, where R_{sch} is the Schwarzschild radius of the black hole. This is the "epoch of merger". At this point, the amplitude of the gravitational wave is maximum, and so is its frequency, which is always twice the orbital frequency. In this problem, we will only focus on the gravitational waves before the merger, when Kepler's laws are assumed to be valid. After the merger, the form of gravitational waves will drastically change.

(T11.1) Consider the observed gravitational waves shown in the figure above. Estimate the time period, T_0, and hence calculate the frequency, f_0, of gravitational waves just before the epoch of merger. **3**

(T11.2) For any main sequence (MS) star, the radius of the star, R_{MS}, and its mass, M_{MS}, are related by a power law given as, **10**

$$\begin{aligned} R_{\text{MS}} &\propto (M_{\text{MS}})^{\alpha} \\ \text{where } \alpha &= 0.8 \qquad \text{for } M_\odot < M_{\text{MS}} \\ &= 1.0 \qquad \text{for } 0.08\, M_\odot \le M_{\text{MS}} \le M_\odot \end{aligned}$$

If the central object were a main sequence star, write an expression for the maximum frequency of the gravitational waves, f_{MS}, in terms of the mass of the star in units of solar masses $(M_{\text{MS}}/M_\odot)$ and α.

(T11.3) Using the above result, determine the appropriate value of α that will give the maximum possible frequency of gravitational waves, $f_{\text{MS,max}}$ for any main sequence star. Evaluate this frequency. **9**

(T11.4) White dwarf (WD) stars have a maximum mass of $1.44\ M_\odot$ (known as the Chandrasekhar limit) and obey the mass-radius relation $R \propto M^{-1/3}$. The radius of a solar mass white dwarf is equal to 6000 km. Find the highest frequency of emitted gravitational waves, $f_{\text{WD,max}}$, if the test mass is orbiting a white dwarf. **8**

(T11.5) Neutron stars (NS) are a peculiar type of compact object, which have masses between 1 and $3\ M_\odot$ and radii in the range $10 - 15$ km. Find the range of frequencies of emitted gravitational waves, $f_{\text{NS,min}}$ and $f_{\text{NS,max}}$, if the test mass is orbiting a neutron star at a distance close to the neutron star radius. **8**

(T11.6) If the test mass is orbiting a black hole (BH), write the expression for the frequency of emitted gravitational waves, f_{BH}, in terms of the mass of the black hole, M_{BH}, and the solar mass $M_\odot$. **7**

(T11.7) Based only on the time period (or frequency) of gravitational waves before the epoch of merger, determine whether the central object can be a main sequence star (MS), a white dwarf (WD), a neutron star (NS), or a black hole (BH). Tick the correct option in the Summary Answersheet. Estimate the mass of this object, M_{obj}, in units of $M_\odot$. **5**

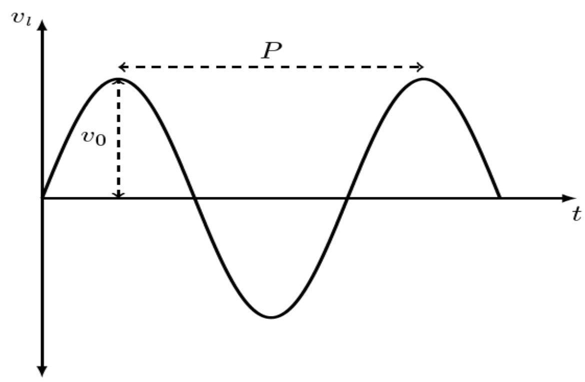
(T12) **Exoplanets**

Two major methods for the detection of exoplanets (planets around stars other than the Sun) are the radial velocity (or so-called "wobble") method and the transit method. In this problem, we find out how a combination of the results of these two methods can reveal a lot of information about an orbiting exoplanet and its host star.

Throughout this problem, we consider the case of a planet of mass M_{p} and radius R_{p} moving in a circular orbit of radius a around a star of mass $M_{\text{s}} (M_{\text{s}} \gg M_{\text{p}})$ and radius R_{s}. The normal to the orbital plane of the planet is inclined at angle i with respect to the line of sight (so $i = 90^\circ$ would mean an "edge on" orbit). We assume that there is no other planet orbiting the star and that $R_{\text{s}} \ll a$.

The "Wobble" Method:
When a planet and a star orbit each other around their barycentre, the star is seen to move slightly, or "wobble", since the centre of mass of the star is not coincident with the barycentre of the star-planet system. As a result, the light received from the star undergoes a small Doppler shift related to the velocity of this wobble.

The line of sight velocity, v_l, of the star can be determined from the Doppler shift of a known spectral line and its periodic variation with time, t, as shown in the schematic diagram below. In the diagram, the two measurable quantities in this method, namely the orbital period P and maximum line of sight velocity v_0 are shown.



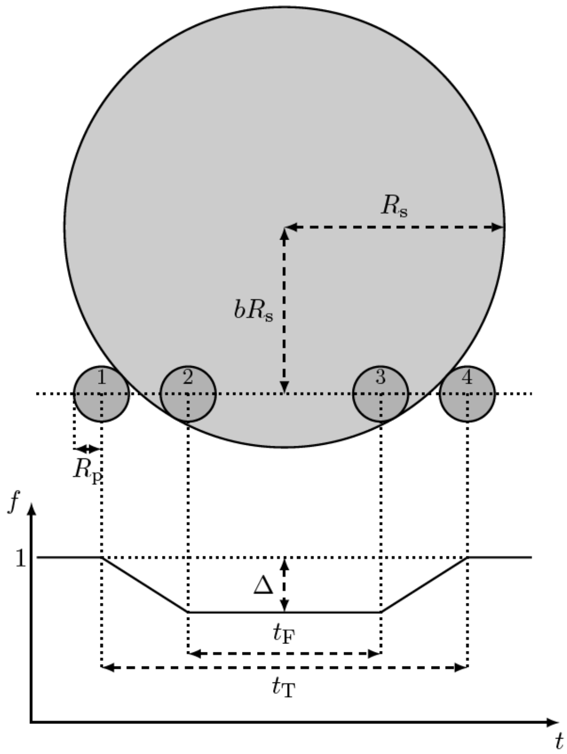
(T12.1) Derive expressions for the orbital radius (a) and orbital speed (v_{p}) of the planet in terms of M_s and P. **3**

(T12.2) Obtain a lower limit for the mass of the planet, $M_{\text{p,min}}$ in terms of M_{s}, v_0 and v_{p}. **4**

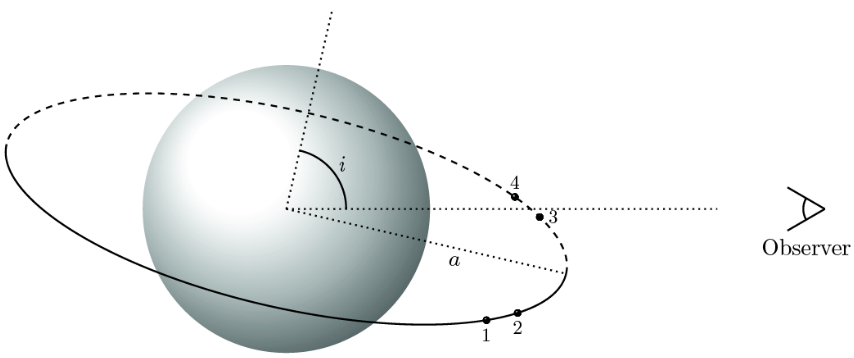
The Transit Method:

As a planet orbits its host star, for orientations of the orbital plane close to "edge-on" ($i \approx 90^{\circ}$), it will pass periodically, or "transit", in front of the stellar disc as seen by the observer. This would cause a tiny decrease in the observed stellar flux, which can be measured. The schematic diagram below (NOT drawn to scale) shows the situation from the observer's perspective and the resulting transit light curve (normalised flux, *f*, vs time, *t*) for a uniformly bright stellar disc.

If the inclination angle i is exactly 90°, the planet would be seen to cross the stellar disc along a diameter. For other values of i, the transit occurs along a chord, whose centre lies at a distance bR_{s} from the centre of the stellar disc, as shown. The no-transit flux is normalised to 1 and the maximum dip during the transit is given by Δ.



The four significant points in the transit are the first, second, third and fourth contacts, marked by the positions 1 to 4, respectively, in the figure above. The time interval during the second and third contacts is denoted as t_{F}, when the disc of the planet overlaps the stellar disc fully. The time interval between the first and fourth contacts is denoted by t_{T}. These points are also marked in the schematic diagram below showing a "side-on" view of the orbit (NOT drawn to scale).



The measurable quantities in the transit method are P, t_{T}, t_{F} and Δ.

(T12.3) Find the constraint on i in terms of R_{s} and a for the transit to be at all visible to the distant observer. **2**

(T12.4) Express Δ in terms of R_{s} and R_{p}. **1**

(T12.5) Express t_{T} and t_{F} in terms of R_{s}, R_{p}, a, P and b. **8**

(T12.6) In the approximation of an orbit much larger than the stellar radius, show that the parameter b is given by **5**

$$b = \left[1 + \Delta - 2\sqrt{\Delta}\frac{1 + \left(\frac{t_{\mathrm{F}}}{t_{\mathrm{T}}}\right)^2}{1 - \left(\frac{t_{\mathrm{F}}}{t_{\mathrm{T}}}\right)^2}\right]^{1/2}$$

(T12.7) Use the result of part (T12.6) to obtain an expression for the ratio a/R_{s} in terms of the measurable transit parameters, using a suitable approximation. **3**

(T12.8) Combine the results of the wobble method and the transit method to determine the stellar mean density $\rho_{\mathrm{s}} = \frac{M_{\mathrm{s}}}{4\pi R_{\mathrm{s}}^3/3}$ in terms of t_{T}, t_{F}, Δ and P. **6**

Rocky or gaseous:

Let us consider an edge-on ($i = 90^{\mathrm{o}}$) star-planet system (with a circular planetary orbit), as seen from the Earth. It is known that the host star has mass $1.00\ M_{\odot}$. Transits are observed with a period (P) of 50.0 days and total transit duration (t_{T}) of 1.00 hour. The transit depth (Δ) is 0.0064. The same system is also observed in the wobble method to have a maximum line of sight velocity of $0.400\ \mathrm{ms}^{-1}$.

(T12.9) Find the orbital radius a of the planet in units of AU and in metres. **2**

(T12.10) Find the ratio $t_{\mathrm{F}}/t_{\mathrm{T}}$ of the system. **2**

(T12.11) Obtain the mass M_{p} and radius R_{p} of the planet in terms of the mass ($M_{\oplus}$) and radius ($R_{\oplus}$) of the Earth respectively. Is the composition of the planet likely to be rocky or gaseous? Tick the box for ROCKY or GASEOUS in the Summary Answersheet. **8**

Transit light curves with starspots and limb darkening:

(T12.12) Consider a planetary transit with $i = 90^{\mathrm{o}}$ around a star which has a starspot on its equator, comparable to the radius of the planet, R_{p}. The rotation period of the star is $2P$. Draw schematic diagrams of the transit light curve for five successive transits of the planet (on the templates provided in the Summary Answersheet). The no-transit flux for each transit may all be normalised to unity, independently. Assume that the planet does not encounter the starspot during the first transit, but does during the second. **4**

(T12.13) Throughout this problem we have considered a uniformly bright stellar disc. However, real stellar discs have limb darkening. Draw a schematic transit light curve when limb darkening is present in the host star.