(D1) **Binary Pulsar**

Through systematic searches during the past decades, astronomers have found a large number of millisecond pulsars (spin period < 10 ms). Majority of these pulsars are found in binaries, with nearly circular orbits.

For a pulsar in a binary orbit, the measured pulsar spin period \(P\) and the measured line-of-sight acceleration \(a\) both vary systematically due to orbital motion. For circular orbits, this variation can be described mathematically in terms of orbital phase \(\phi\) (\(0 \leq \phi \leq 2\pi\)) as,

\[
P(\phi) = P_0 + P_t \cos \phi
\]

\[
a(\phi) = -a_t \sin \phi
\]

where

\[
P_t = \frac{2\pi P_0 r}{c P_B}
\]

\[
a_t = \frac{4\pi^2 r}{P_B^2}
\]

where \(P_B\) is the orbital period of the binary, \(P_0\) is the intrinsic spin period of the pulsar and \(r\) is the radius of the orbit.

The following table gives one such set of measurements of \(P\) and \(a\) at different heliocentric epochs, \(T\), expressed in truncated Modified Julian Days (tMJD), i.e. number of days since MJD = 2,440,000.

<table>
<thead>
<tr>
<th>No.</th>
<th>(t\text{MJD})</th>
<th>(P) ((\mu\text{s}))</th>
<th>(a) ((\text{m s}^{-2}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5740.654</td>
<td>7587.8889</td>
<td>-0.92 ± 0.08</td>
</tr>
<tr>
<td>2</td>
<td>5740.703</td>
<td>7587.8334</td>
<td>-0.24 ± 0.08</td>
</tr>
<tr>
<td>3</td>
<td>5746.100</td>
<td>7588.4100</td>
<td>-1.68 ± 0.04</td>
</tr>
<tr>
<td>4</td>
<td>5746.675</td>
<td>7588.5810</td>
<td>+1.67 ± 0.06</td>
</tr>
<tr>
<td>5</td>
<td>5981.811</td>
<td>7587.8836</td>
<td>+0.72 ± 0.06</td>
</tr>
<tr>
<td>6</td>
<td>5983.932</td>
<td>7587.8552</td>
<td>-0.44 ± 0.08</td>
</tr>
<tr>
<td>7</td>
<td>6005.893</td>
<td>7589.1029</td>
<td>+0.52 ± 0.08</td>
</tr>
<tr>
<td>8</td>
<td>6040.857</td>
<td>7589.1350</td>
<td>+0.00 ± 0.04</td>
</tr>
<tr>
<td>9</td>
<td>6335.904</td>
<td>7589.1358</td>
<td>+0.00 ± 0.02</td>
</tr>
</tbody>
</table>

By plotting \(a(\phi)\) as a function of \(P(\phi)\), we can obtain a parametric curve. As evident from the relations above, this curve in the period-acceleration plane is an ellipse.

In this problem, we estimate the intrinsic spin period, \(P_0\), the orbital period, \(P_B\), and the orbital radius, \(r\), by an analysis of this data set, assuming a circular orbit.

(D1.1) Plot the data, including error bars, in the period-acceleration plane (mark your graph as "D1.1").

**Solution:**

Graph Number : D1.1
• Plot uses more than 50% of graph paper: 0.5
• Axes labels ($P$ and $a$): 0.5
• Dimensions of axes: 0.5
• Ticks and values on axes (or scale written explicitly): 0.5
• Points correctly plotted:

<table>
<thead>
<tr>
<th>Points plotted</th>
<th>9</th>
<th>8</th>
<th>7</th>
<th>6</th>
<th>5</th>
<th>&lt; 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Marks given</td>
<td>4.0</td>
<td>3.5</td>
<td>3.0</td>
<td>2.0</td>
<td>1.0</td>
<td>0</td>
</tr>
</tbody>
</table>

Correctness of points: deduction of 0.5 for each wrong point.
• Errorbars on points (at least 5): 1.0

(D1.2) Draw an ellipse that appears to be a best fit to the data (on the same graph “D1.1”).

Solution:
See above
• Elliptical curve with visual best fit: 1.0
• Curve symmetric about $a = 0$ line: 0.5
- Curve symmetric about some value of $P$ ($P \approx 7588.48$): 0.5

(D1.3) From the plot, estimate $P_0$, $P_t$ and $a_t$, including error margins.

Solution:
Values are determined from lengths of axes of ellipse and mid-point of $P$-axis. Error margins may be determined by estimating extreme ellipses covering the points with errorbars. Any reasonable method of estimating error margins will be accepted.

\[
2P_t = (1.34 \pm 0.04) \mu s \quad \text{[} (13.4 \pm 0.4) \text{ cm on graph}] \\
P_t = (0.67 \pm 0.02) \mu s \\
2P_0 = (7588.48 \pm 0.02) \mu s \\
P_0 = (7588.38 \pm 0.02) \mu s \\
2a_t = (3.42 \pm 0.12) \text{ m s}^{-2} \quad \text{[} (17.1 \pm 0.6) \text{ cm on graph}] \\
a_t = (1.71 \pm 0.06) \text{ m s}^{-2}
\]

- Marking table:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Half credit Minimum</th>
<th>Full credit Minimum</th>
<th>Maximum</th>
<th>Half credit Maximum</th>
<th>Maximum</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_t(\mu s)$</td>
<td>0.59</td>
<td>0.63</td>
<td>0.71</td>
<td>0.75</td>
<td>0.05</td>
</tr>
<tr>
<td>$\delta P_t(\mu s)$</td>
<td>0.01</td>
<td>0.02</td>
<td>0.04</td>
<td>0.05</td>
<td></td>
</tr>
<tr>
<td>$P_0(\mu s)$</td>
<td>7588.38</td>
<td>7588.43</td>
<td>7588.53</td>
<td>7588.58</td>
<td></td>
</tr>
<tr>
<td>$\delta P_0(\mu s)$</td>
<td>0.01</td>
<td>0.02</td>
<td>0.04</td>
<td>0.05</td>
<td></td>
</tr>
<tr>
<td>$a_t(\text{m s}^{-2})$</td>
<td>1.61</td>
<td>1.65</td>
<td>1.77</td>
<td>1.81</td>
<td></td>
</tr>
<tr>
<td>$\delta a_t(\text{m s}^{-2})$</td>
<td>0.04</td>
<td>0.05</td>
<td>0.07</td>
<td>0.08</td>
<td></td>
</tr>
</tbody>
</table>

- Wrong values due to wrong/poor plot/fit in (D1.1) and (D1.2) WILL BE penalised.
- Error estimation is based on graph drawing. Quoted values correspond to the envelope of possible ellipses drawn to include all points with errorbars. Any reasonable method to estimate error to be given credit.

(D1.4) Write expressions for $P_B$ and $r$ in terms of $P_0$, $P_t$, $a_t$.

Solution:
We can easily recover the orbital period ($P_B$) and the radius of the orbit ($r$) in a circular orbit:

\[
a_t = 4\pi^2 \frac{r}{P_B^2} \\
\therefore r = \frac{P_B^2 a_t}{4\pi^2} \\
P_t = \frac{2\pi P_0}{P_B} \times \frac{r}{c} \\
= \frac{2\pi P_0}{P_B} \times \frac{P_B^2 a_t}{4\pi^2} = \frac{P_0 P_B a_t}{2\pi c}
\]
\[
\therefore P_b = \frac{P_t^2 2\pi c}{P_0^2 a_t}
\]
\[
r = \frac{a_t}{4\pi^2} \left( \frac{P_t 2\pi c}{P_0 a_t} \right)^2
\]
\[
\therefore r = \left( \frac{P_t}{P_0} \right)^2 \frac{c^2}{a_t}
\]

Alternative algebraic routes accepted. Each of \(P_b\) and \(r\) carry 2.0 marks.

(D1.5) Calculate approximate value of \(P_b\) and \(r\) based on your estimations made in (D1.3), including error margins.

Solution:
\[
P_b = \frac{P_t^2 2\pi c}{P_0^2 a_t}
\]
\[
= \frac{0.67}{7588.48} \times \frac{2\pi \times 2.998 \times 10^8}{1.71} s
\]
\[
= 96260 s = 1.12570 d
\]
\[
\Delta P_b = P_b \sqrt{\left( \frac{\Delta P_t}{P_t} \right)^2 + \left( \frac{\Delta P_0}{P_0} \right)^2 + \left( \frac{\Delta a_t}{a_t} \right)^2}
\]
\[
= 1.12570 \times \sqrt{\left( \frac{0.02}{0.67} \right)^2 + \left( \frac{0.02}{7588.48} \right)^2 + \left( \frac{0.06}{1.71} \right)^2} d
\]
\[
= 1.12570 \times 0.0461 \simeq 0.052 d
\]
\[
\therefore P_b = (1.13 \pm 0.05) d
\]
\[
r = \left( \frac{P_t}{P_0} \right)^2 \frac{c^2}{a_t}
\]
\[
= \left( \frac{0.67}{7588.48} \right)^2 \times \frac{(2.998 \times 10^8)^2}{1.71} m
\]
\[
\therefore r = 4.09739 \times 10^8 m = 2.73890 \times 10^{-3} AU
\]
\[
\Delta r = r \sqrt{\left( \frac{2\Delta P_t}{P_t} \right)^2 + \left( \frac{2\Delta P_0}{P_0} \right)^2 + \left( \frac{\Delta a_t}{a_t} \right)^2}
\]
\[
= 2.73890 \times 10^{-3} \times \sqrt{\left( \frac{2 \times 0.02}{0.67} \right)^2 + \left( \frac{2 \times 0.02}{7588.48} \right)^2 + \left( \frac{0.06}{1.71} \right)^2} AU
\]
\[
= 2.73890 \times 10^{-3} \times 0.06925 AU \simeq 0.19 \times 10^{-3} AU
\]
\[
r = (2.74 \pm 0.19) \times 10^{-3} AU
\]

Errors in \(P_b\) and \(r\) can be also estimated as maximum possible (worst case) error. In such case, errors would be about 1.5 times the standard error calculated above (\(\delta P_b = 0.07, \delta r = 0.25\)).

(D1.6) Calculate orbital phase, \(\phi\), corresponding to the epochs of the following five observations.
in the above table: data rows 1, 4, 6, 8, 9.

Solution:
Using these newly determined orbital parameters, we can calculate the angular orbital phase for each data point, i.e., for each pair of acceleration and period measured \((P, a)\).

\[
\phi = \tan^{-1}\left(\frac{a}{a_i P - P_0}\right)
\]

Care has to be taken to choose the value of the phase from among \(\phi, \pi \pm \phi, 2\pi - \phi\), depending on the sign of \(\cos \phi\) and \(\sin \phi\).

<table>
<thead>
<tr>
<th>Sr. no.</th>
<th>(T) (tMJD)</th>
<th>(P) (µs)</th>
<th>(a) (m/s(^{-2}))</th>
<th>(\phi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5740.654</td>
<td>7587.8889</td>
<td>-0.92</td>
<td>148.62(^{\circ})</td>
</tr>
<tr>
<td>4</td>
<td>5746.675</td>
<td>7588.5810</td>
<td>+1.67</td>
<td>278.77(^{\circ})</td>
</tr>
<tr>
<td>6</td>
<td>5983.932</td>
<td>7587.8552</td>
<td>-0.44</td>
<td>164.57(^{\circ})</td>
</tr>
<tr>
<td>8</td>
<td>6040.857</td>
<td>7589.1350</td>
<td>+0.00</td>
<td>0.00(^{\circ})</td>
</tr>
<tr>
<td>9</td>
<td>6335.904</td>
<td>7589.1358</td>
<td>+0.00</td>
<td>0.00(^{\circ})</td>
</tr>
</tbody>
</table>

- Credit for each correct value: 1.0 for first three, 0.5 for last two.
- Credit for \(\pi \pm \phi\) or \(2\pi - \phi\) is 0.5 per value.
- All values wrong due to wrong expression for \(\phi\) gets a maximum of 1.0 mark.
- Values in radians accepted.

(D1.7) Refine the estimate of the orbital period, \(P_B\), using the results in part (D1.6) in the following way:

(D1.7a) First determine the initial epoch, \(T_0\), which corresponds to the nearest epoch of zero phase before the first observation.

Solution:
\[
\frac{T_1 - T_0}{P_B} = \frac{\phi_1}{2\pi} \Rightarrow T_0 = T_1 - \frac{\phi_1}{2\pi}P_B
\]

\[
T_0 = 5740.654 - \frac{148.62^{\circ}}{360^{\circ}} \times 1.12570 \text{tMJD}
\]

\[
T_0 = 5740.189 \text{tMJD}
\]

Tolerance: \(\pm 0.002 \text{tMJD}\). Using \(P_0\) instead of \(P_B\) gets zero.

(D1.7b) The expected time, \(T_{\text{calc}}\), of the estimated phase of each observation is given by

\[
T_{\text{calc}} = T_0 + \left( n + \frac{\phi}{360^{\circ}} \right)P_B,
\]

where \(n\) is the number of full cycle of orbital phases elapsed between \(T_0\) and \(T_{\text{calc}}\). Estimate \(n\) and \(T_{\text{calc}}\) for each of the five observations in part (D1.6). Note down difference \(T_{O-C}\) between observed \(T\) and \(T_{\text{calc}}\). Enter these calculations in the table given in the Summary Answersheet.

Solution:
Data Analysis Examination

\[ T_{\text{calc}} = T_0 + \left( n + \frac{\phi}{360^\circ} \right) P_B \]

where \( n = \text{Integer part of} \left[ \frac{(T - T_0)}{P_B} \right] \).

<table>
<thead>
<tr>
<th>Sr. No.</th>
<th>( T ) (tMJD)</th>
<th>( \phi )</th>
<th>( n )</th>
<th>( T_{\text{calc}} ) (MJD)</th>
<th>( T_{\text{O-C}} ) (days)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5740.654</td>
<td>148.62°</td>
<td>0</td>
<td>5740.654</td>
<td>0.000</td>
</tr>
<tr>
<td>4</td>
<td>5746.675</td>
<td>278.77°</td>
<td>5</td>
<td>5746.689</td>
<td>−0.014</td>
</tr>
<tr>
<td>6</td>
<td>5983.932</td>
<td>164.57°</td>
<td>216</td>
<td>5983.855</td>
<td>0.077</td>
</tr>
<tr>
<td>8</td>
<td>6040.857</td>
<td>0.00°</td>
<td>267</td>
<td>6040.751</td>
<td>0.106</td>
</tr>
<tr>
<td>9</td>
<td>6335.904</td>
<td>0.00°</td>
<td>529</td>
<td>6335.684</td>
<td>0.220</td>
</tr>
</tbody>
</table>

Deduction for each wrong/missing value of \( n, T_{\text{calc}} \) and \( T_{\text{O-C}} \): 0.5

No double penalty in one row.

(D1.7c) Plot \( T_{\text{O-C}} \) against \( n \) (mark your graph as “D1.7”).

Solution:

Graph Number : D1.7

- Plot uses more than 50% of graph paper: 0.5
- Axes labels \( (T_{\text{O-C}} \) and \( n \) including dimensions: 0.5
- Ticks and values on axes (or scale written explicitly): 0.5
- Points correctly plotted: 0.5 for each point
- Goodness of linear fit credited in next part

(D1.7d) Determine the refined values of the initial epoch, \( T_{0,r} \), and the orbital period, \( P_{B,r} \).
Solution:
A linear fit to the plot of $T_{O-C}$ vs $n$ gives the offset of period per cycle (slope) and the shift in the zero-phase point (intercept).
This concept, which may be evident in the subsequent calculation, gains the credit, explicit statement is not necessary.
From a linear fit, 
\[
\text{Slope} = 0.00043 \text{d/n} \quad \text{Intercept} = -0.010 \text{d}
\]
- Credit for good visual linear fit: 1.0
- Correct values of slope and intercept: 1.0 each
- Tolerance: $\pm 0.00002$ in slope and $\pm 0.002$ in intercept.

\[
T_{0,r} = 5740.189 - 0.010 = 5740.179 \text{tMJD}
\]
\[
P_{B,r} = (1.12570 + 0.00043) \text{d}
= 1.12613 \text{d}
\]
\[
P_B = 1.1261 \text{d}
\]
Incorrect sign of correction applied carries penalty of 0.5 for each quantity.
(D2) **Distance to the Moon**

Geocentric ephemerides of the Moon for September 2015 are given in the form of a table. Each reading was taken at 00:00 UT.

<table>
<thead>
<tr>
<th>Date</th>
<th>R.A. (α)</th>
<th>Dec. (δ)</th>
<th>Angular Size (θ)</th>
<th>Phase (φ)</th>
<th>Elongation of Moon</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>h m s</td>
<td>o ′ ′′</td>
<td>″</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sep 01</td>
<td>0 36 46</td>
<td>3 6 16.8</td>
<td>1991.2</td>
<td>0.927</td>
<td>148.6° W</td>
</tr>
<tr>
<td>Sep 02</td>
<td>1 33 51.34</td>
<td>7 32 26.1</td>
<td>1974.0</td>
<td>0.852</td>
<td>134.7° W</td>
</tr>
<tr>
<td>Sep 03</td>
<td>2 30 45.03</td>
<td>11 25 31.1</td>
<td>1950.7</td>
<td>0.759</td>
<td>121.1° W</td>
</tr>
<tr>
<td>Sep 04</td>
<td>3 27 28.48</td>
<td>14 32 4.3</td>
<td>1923.9</td>
<td>0.655</td>
<td>107.9° W</td>
</tr>
<tr>
<td>Sep 05</td>
<td>4 23 52.28</td>
<td>16 43 18.2</td>
<td>1896.3</td>
<td>0.546</td>
<td>95.2° W</td>
</tr>
<tr>
<td>Sep 06</td>
<td>5 19 37.25</td>
<td>17 55 4.4</td>
<td>1869.8</td>
<td>0.438</td>
<td>82.8° W</td>
</tr>
<tr>
<td>Sep 07</td>
<td>6 14 19.23</td>
<td>18 7 26.6</td>
<td>1845.5</td>
<td>0.336</td>
<td>70.7° W</td>
</tr>
<tr>
<td>Sep 08</td>
<td>7 7 35.58</td>
<td>17 23 55.6</td>
<td>1824.3</td>
<td>0.243</td>
<td>59.0° W</td>
</tr>
<tr>
<td>Sep 09</td>
<td>7 59 11.04</td>
<td>15 50 33.0</td>
<td>1806.5</td>
<td>0.163</td>
<td>47.5° W</td>
</tr>
<tr>
<td>Sep 10</td>
<td>8 49 0.93</td>
<td>13 34 55.6</td>
<td>1792.0</td>
<td>0.097</td>
<td>36.2° W</td>
</tr>
<tr>
<td>Sep 11</td>
<td>9 37 11.42</td>
<td>10 45 27.7</td>
<td>1780.6</td>
<td>0.047</td>
<td>25.1° W</td>
</tr>
<tr>
<td>Sep 12</td>
<td>10 23 57.77</td>
<td>7 30 47.7</td>
<td>1772.2</td>
<td>0.015</td>
<td>14.1° W</td>
</tr>
<tr>
<td>Sep 13</td>
<td>11 9 41.86</td>
<td>3 59 28.8</td>
<td>1766.5</td>
<td>0.001</td>
<td>3.3° W</td>
</tr>
<tr>
<td>Sep 14</td>
<td>11 54 49.80</td>
<td>0 19 50.2</td>
<td>1763.7</td>
<td>0.005</td>
<td>7.8° E</td>
</tr>
<tr>
<td>Sep 15</td>
<td>12 39 50.01</td>
<td>-3 20 3.7</td>
<td>1763.8</td>
<td>0.026</td>
<td>18.6° E</td>
</tr>
<tr>
<td>Sep 16</td>
<td>13 25 11.64</td>
<td>-6 52 18.8</td>
<td>1767.0</td>
<td>0.065</td>
<td>29.5° E</td>
</tr>
<tr>
<td>Sep 17</td>
<td>14 11 23.13</td>
<td>-10 9 4.4</td>
<td>1773.8</td>
<td>0.120</td>
<td>40.4° E</td>
</tr>
<tr>
<td>Sep 18</td>
<td>14 58 50.47</td>
<td>-13 2 24.7</td>
<td>1784.6</td>
<td>0.189</td>
<td>51.4° E</td>
</tr>
<tr>
<td>Sep 19</td>
<td>15 47 54.94</td>
<td>-15 24 14.6</td>
<td>1799.6</td>
<td>0.270</td>
<td>62.5° E</td>
</tr>
<tr>
<td>Sep 20</td>
<td>16 38 50.31</td>
<td>-17 6 22.8</td>
<td>1819.1</td>
<td>0.363</td>
<td>73.9° E</td>
</tr>
<tr>
<td>Sep 21</td>
<td>17 31 40.04</td>
<td>-18 0 52.3</td>
<td>1843.0</td>
<td>0.463</td>
<td>85.6° E</td>
</tr>
<tr>
<td>Sep 22</td>
<td>18 26 15.63</td>
<td>-18 0 41.7</td>
<td>1870.6</td>
<td>0.567</td>
<td>97.6° E</td>
</tr>
<tr>
<td>Sep 23</td>
<td>19 22 17.51</td>
<td>-17 0 50.6</td>
<td>1900.9</td>
<td>0.672</td>
<td>110.0° E</td>
</tr>
<tr>
<td>Sep 24</td>
<td>20 19 19.45</td>
<td>-14 59 38.0</td>
<td>1931.9</td>
<td>0.772</td>
<td>122.8° E</td>
</tr>
<tr>
<td>Sep 25</td>
<td>21 16 55.43</td>
<td>-11 59 59.6</td>
<td>1961.1</td>
<td>0.861</td>
<td>136.2° E</td>
</tr>
<tr>
<td>Sep 26</td>
<td>22 14 46.33</td>
<td>-8 10 18.3</td>
<td>1985.5</td>
<td>0.933</td>
<td>150.0° E</td>
</tr>
<tr>
<td>Sep 27</td>
<td>23 12 43.63</td>
<td>-3 44 28.7</td>
<td>2002.0</td>
<td>0.981</td>
<td>164.0° E</td>
</tr>
<tr>
<td>Sep 28</td>
<td>0 10 48.32</td>
<td>0 58 58.2</td>
<td>2008.3</td>
<td>1.000</td>
<td>178.3° E</td>
</tr>
<tr>
<td>Sep 29</td>
<td>1 9 5.89</td>
<td>5 38 54.3</td>
<td>2003.6</td>
<td>0.988</td>
<td>167.4° W</td>
</tr>
<tr>
<td>Sep 30</td>
<td>2 7 39.02</td>
<td>9 54 16.1</td>
<td>1988.4</td>
<td>0.947</td>
<td>153.2° W</td>
</tr>
</tbody>
</table>
The composite graphic\(^1\) below shows multiple snapshots of the Moon taken at different times during the total lunar eclipse, which occurred in this month. For each shot, the centre of frame was coinciding with the central north-south line of umbra.

For this problem, assume that the observer is at the centre of the Earth and angular size refers to angular diameter of the relevant object / shadow.

(D2.1) In September 2015, apogee of the lunar orbit is closest to New Moon / First Quarter / Full Moon / Third Quarter.

Tick the correct answer in the Summary Answer sheet. No justification for your answer is necessary.

**Solution:**
From the table we see that the angular size of Moon is smallest close to the New Moon day. Thus, the answer is New Moon.

**Justification is NOT necessary for full credit.**

(D2.2) In September 2015, the ascending node of lunar orbit with respect to the ecliptic is closest to New Moon / First Quarter / Full Moon / Third Quarter.

Tick the correct answer in the Summary Answer sheet. No justification for your answer is necessary.

**Solution:**
As there is an eclipse happening in this month, the lunar nodes are close to Full Moon day and New Moon day. Next we notice that lowest declination of Moon is just 18\(^{\circ}\). This means that after the New Moon day, the orbit of Moon is above the ecliptic. In other words, the ascending node is near the New Moon.

**Justification is NOT necessary for full credit.**

(D2.3) Estimate the eccentricity, \(e\), of the lunar orbit from the given data.

**Solution:**
The largest angular size of the Moon in the ephemerides is 2008.3\(^{\prime}\) and the smallest angular size is 1763.7\(^{\prime}\). The distance is inversely proportional to the angular size. Hence ratio of distance at perigee to the distance at apogee is:

\[
\text{Ratio} = \frac{r_{\text{perigee}}}{r_{\text{apogee}}} = \frac{\mathcal{G} \Delta}{1 + e} \times \frac{1 - e}{\mathcal{G} \Delta}
\]

\(^1\)Credit: NASA’s Scientific Visualization Studio
\[
\therefore \frac{1 - e}{1 + e} = \frac{1763.7}{2008.3} = 0.87821
\]
\[
\therefore e = \frac{1 - 0.87821}{1 + 0.87821} = 0.064846
\]
\[
e \approx 0.065
\]

Rounding off is done to account for the fact that our data are not continuous, hence exact angular sizes at perigee and apogee are not known. A non-rounded answer will also receive full credit.

(D2.4) Estimate the angular size of the umbra, $\theta_{\text{umbra}}$, in terms of the angular size of the Moon, $\theta_{\text{Moon}}$. Show your working on the image given on the backside of the Summary Answersheet.

Solution:
The following construction is shown.

Only two chords are necessary to determine the centre.

The credit is divided in two parts for the drawing:

- Realisation that centre of umbra circle needs to be determined to find $\theta_{\text{umbra}}$: 1.5
- Accurate determination of centre of umbra circle by geometric construction: 2.5
  Determination of centre of hand-drawn circle: maximum 1.5
Measuring diameters of umbra and Moon: 0.5 each

By estimating approximate centre of the shadow in the image, we find out,

\[
\frac{\theta_{\text{umbra}}}{\theta_{\text{Moon}}} = \frac{d_{\text{umbra}}}{d_{\text{Moon}}} = \frac{9.1}{3.3} = 2.76
\]

\[\therefore \theta_{\text{umbra}} \approx 2.76\theta_{\text{Moon}}\]

Acceptable range: ±0.10.

The angle subtended by the Sun at Earth on the day of the lunar eclipse is known to be \(\theta_{\text{Sun}} = 1915.0''\). In the figure below, \(S_1R_1\) and \(S_2R_2\) are rays coming from diametrically opposite ends of the solar disk. The figure is not to scale.

Calculate the angular size of the penumbra, \(\theta_{\text{penumbra}}\), in terms of \(\theta_{\text{Moon}}\). Assume the observer to be at the centre of the Earth.

**Solution:**

The following diagram needs to be drawn.

Angular size of umbra is \(\theta_{\text{umbra}} = 2\angle BOQ\)

Angular size of penumbra is \(\theta_{\text{penumbra}} = 2\angle AOQ\)

We have

\[QA = QP + PA = OE + PE \tan \theta_1\]
\[ \approx R_\odot + d_{\text{Moon}}\theta_1 \quad \text{since } PA \ll PE \]
\[ \approx R_\odot + d_{\text{Moon}}\frac{\theta_{\text{Sun}}}{2} \quad \text{since } \theta_1 \approx \theta_2 \approx \theta_{\text{Sun}}/2 \]

and

\[ QB =QP-PB \]
\[ =OE-PE\tan \theta_2 \]
\[ \approx R_\odot - d_{\text{Moon}}\theta_2 \]
\[ \approx R_\odot - d_{\text{Moon}}\frac{\theta_{\text{Sun}}}{2} \]

\[ . \quad . \quad \therefore \theta_{\text{penumbra}} = 2\angle AOQ = 2\tan^{-1}\left(\frac{QA}{OQ}\right) \approx 2\frac{QA}{OQ} \quad \text{since } QA \ll OQ \]
\[ = 2\frac{R_\odot + d_{\text{Moon}}\frac{\theta_{\text{Sun}}}{2}}{d_{\text{Moon}}} = \frac{2R_\odot}{d_{\text{Moon}}} + \theta_{\text{Sun}} \]

and

\[ \theta_{\text{umbra}} = 2\angle BOQ = 2\tan^{-1}\left(\frac{QB}{OQ}\right) \approx 2\frac{QB}{OQ} \]
\[ = 2\frac{R_\odot - d_{\text{Moon}}\frac{\theta_{\text{Sun}}}{2}}{d_{\text{Moon}}} = \frac{2R_\odot}{d_{\text{Moon}}} - \theta_{\text{Sun}} \]

Subtracting,
\[ \theta_{\text{penumbra}} - \theta_{\text{umbra}} = 2\theta_{\text{Sun}} \Rightarrow \theta_{\text{penumbra}} = \theta_{\text{umbra}} + 2\theta_{\text{Sun}} \]

We have,
\[ \theta_{\text{umbra}} = 2.76\theta_{\text{Moon}} \quad \text{and} \quad \theta_{\text{Sun}} = 1915.0'' \]

From the given data, \( \theta_{\text{Moon}} = 2008.3'' \).

Therefore,
\[ \theta_{\text{penumbra}} = 2.76\theta_{\text{Moon}} + 2\frac{1915.0}{2008.3}\theta_{\text{Moon}} \]

\[ \theta_{\text{penumbra}} = 4.67\theta_{\text{Moon}} \]

Acceptable range: \( 4.57\theta_{\text{Moon}} \) to \( 4.77\theta_{\text{Moon}} \).

Alternative solution:

In the figure below, rays \( HEA \) and \( IFC \) are coming from one edge of solar disk and rays \( HFD \) and \( GEB \) are coming from the opposite edge. The observer (\( O \)) is assumed to be at the centre of the Earth. The Moon travels along the path \( ABCD \) during the course of eclipse.
From figure,

\[ \angle AEB = \angle GEH = \angle HFI = \angle DFC = \angle EJF = \theta_{\text{Sun}} \]

\[ \theta_{\text{umbra}} = \angle BOC = 2.76\theta_{\text{Moon}} \]

\[ \theta_{\text{penumbra}} = \angle AOD \]

\[ \angle AOD = \angle AOB + \angle BOC + \angle COD \]

\[ \angle AOB = \angle AEB \]

\[ \angle COD = \angle CFD \]

\[ \theta_{\text{penumbra}} \approx \angle AEB + \theta_{\text{umbra}} + \angle CFD \]

\[ = 2\theta_{\text{Sun}} + 2.76\theta_{\text{Moon}} = 2 \times 1915.0'' + 2.76 \times 2008.3'' \]

\[ \theta_{\text{penumbra}} = 9372.9'' = 4.67\theta_{\text{Moon}} \]

(D2.6) Let \( \theta_{\text{Earth}} \) be angular size of the Earth as seen from the centre of the Moon. Calculate the angular size of the Moon, \( \theta_{\text{Moon}} \), as would be seen from the centre of the Earth on the eclipse day in terms of \( \theta_{\text{Earth}} \).

**Solution:**

From the Moon,

\[ \theta_{\text{Earth}} = \frac{2R_E}{d_{\text{Moon}}} \]
From part (D2.5),
\[ \theta_{\text{umbra}} + \theta_{\text{penumbra}} = 2\theta_{\text{Earth}} \]
\[ \therefore \theta_{\text{Earth}} = \frac{\theta_{\text{umbra}} + \theta_{\text{penumbra}}}{2} = \frac{2.76 + 4.67}{2} = 3.72 \theta_{\text{Moon}} \]
\[ \theta_{\text{Moon}} = 0.269 \theta_{\text{Earth}} \]

**Alternative solution:**
Let us say that the Moon is at position of B. Thus, angular size of Earth as seen from this position will be, (see figure in the previous part)
\[ \theta_{\text{Earth}} = \angle EBF = \angle BFD \]
\[ = \angle BFC + \angle CFD \]
\[ \simeq \theta_{\text{umbra}} + \theta_{\text{Sun}} \]

The angular size of the Full Moon on 28 September as seen in the table is 2008.3''.
\[ \theta_{\text{Earth}} = 2.76 \times 2008.3'' + 1915.0'' = 7453.0'' \]
\[ \theta_{\text{Moon}} = 0.269 \theta_{\text{Earth}} = \frac{\theta_{\text{Earth}}}{3.72} \]

(D2.7) Estimate the radius of the Moon, \( R_{\text{Moon}} \), in km from the results above.

**Solution:**
Thus, the radius of Moon will be,
\[ R_{\text{Moon}} = \frac{R_{\oplus}}{3.72} \]
\[ R_{\text{Moon}} = \frac{6371}{3.72} \]
\[ R_{\text{Moon}} \simeq 1713 \text{ km} \]

Acceptable range: \( \pm 20 \text{ km} \).

(D2.8) Estimate the shortest distance, \( r_{\text{perigee}} \), and the farthest distance, \( r_{\text{apogee}} \), to the Moon.

**Solution:**
The shortest and longest distances will be,
\[ r_{\text{perigee}} = \frac{2 \times 1713 \times 206265}{2008.3} \]
\[ r_{\text{perigee}} = 3.52 \times 10^8 \text{ km} \]
\[ r_{\text{apogee}} = \frac{2 \times 1713 \times 206265}{1763.7} \]
\[ r_{\text{apogee}} = 4.01 \times 10^8 \text{ km} \]

(D2.9) Use appropriate data from September 10 to estimate the distance, \( d_{\text{Sun}} \), to the Sun from the Earth.
Solution:

![Diagram showing Earth, Moon, Sun, and Moon's orbit with angles and distances labeled.]

On September 10, phase of Moon is 0.097 and elongation of Moon is 36.2°. Angular size of the Moon on this day is 1792.0". Therefore, distance to Moon (from Earth) on September 10 is

\[ d_{\text{Moon,10}} = \frac{2 \times 1713 \times 206265}{1792.0} = 3.94 \times 10^5 \text{ km} \]

Let \( \angle EMS = \phi_M \)
\( \angle ESM = \phi_S \)
\( \angle SEM = \phi_E \)

\[ \therefore \phi_E = 36.2^\circ \]

\[ \text{phase} = \frac{1 + \cos \phi_M}{2} \]

\[ \therefore \phi_M = \cos^{-1} \left( 2 \times \text{phase} - 1 \right) = \cos^{-1} \left( 2 \times 0.097 - 1 \right) = \cos^{-1}(-0.806) = 143.71^\circ \]

\[ \phi_S = 180^\circ - \phi_E - \phi_M \]
\[ = 180^\circ - 36.2^\circ - 143.71^\circ = 0.09^\circ \]

Now using sine rule,

\[ \frac{d_{\text{Sun}}}{d_{\text{Moon,10}}} = \frac{\sin \phi_M}{\sin \phi_S} \]

\[ \therefore d_{\text{Sun}} = \frac{3.94 \times 10^8 \times \sin 143.71^\circ}{\sin 0.09^\circ} \]

\[ d_{\text{Sun}} = 1.48 \times 10^{11} \text{ m} \]
Type IA Supernovae

Supernovae of type Ia are considered very important for the measurements of large extragalactic distances. The brightening and subsequent dimming of these explosions follow a characteristic light curve, which helps in identifying these as supernovae of type Ia.

Light curves of all type Ia supernovae can be fit to the same model light curve, when they are scaled appropriately. In order to achieve this, we first have to express the light curves in the reference frame of the host galaxy by taking care of the cosmological stretching/dilation of all observed time intervals, $\Delta t_{\text{obs}}$, by a factor of $(1 + z)$. The time interval in the rest frame of the host galaxy is denoted by $\Delta t_{\text{gal}}$.

The rest frame light curve of a supernova changes by two magnitudes compared to the peak in a time interval $\Delta t_0$ after the peak. If we further scale the time intervals by a factor of $s$ (i.e. $\Delta t_s = s\Delta t_{\text{gal}}$) such that the scaled value of $\Delta t_0$ is the same for all supernovae, the light curves turn out to have the same shape. It also turns out that $s$ is related linearly to the absolute magnitude, $M_{\text{peak}}$, at the peak luminosity for the supernova. That is, we can write

$$s = a + bM_{\text{peak}},$$

where $a$ and $b$ are constants. Knowing the scaling factor, one can determine absolute magnitudes of supernovae at unknown distances from the above linear equation.

The table below contains data for three supernovae, including their distance moduli, $\mu$ (for the first two), their recession speed, $cz$, and their apparent magnitudes, $m_{\text{obs}}$, at different times. The time $\Delta t_{\text{obs}} \equiv t - t_{\text{peak}}$ shows number of days from the date at which the respective supernova reached peak brightness. The observed magnitudes have already been corrected for interstellar as well as atmospheric extinction.

<table>
<thead>
<tr>
<th>Name</th>
<th>SN2006TD</th>
<th>SN2006IS</th>
<th>SN2005LZ</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu$ (mag)</td>
<td>34.27</td>
<td>35.64</td>
<td></td>
</tr>
<tr>
<td>$cz$ (km s$^{-1}$)</td>
<td>4515</td>
<td>9426</td>
<td>12060</td>
</tr>
<tr>
<td>$\Delta t_{\text{obs}}$ (days)</td>
<td>$m_{\text{obs}}$ (mag)</td>
<td>$m_{\text{obs}}$ (mag)</td>
<td>$m_{\text{obs}}$ (mag)</td>
</tr>
<tr>
<td>$-15.00$</td>
<td>19.41</td>
<td>18.35</td>
<td>20.18</td>
</tr>
<tr>
<td>$-10.00$</td>
<td>17.48</td>
<td>17.26</td>
<td>18.79</td>
</tr>
<tr>
<td>$-5.00$</td>
<td>16.12</td>
<td>16.42</td>
<td>17.85</td>
</tr>
<tr>
<td>$0.00$</td>
<td>15.74</td>
<td>16.17</td>
<td>17.58</td>
</tr>
<tr>
<td>$5.00$</td>
<td>16.06</td>
<td>16.41</td>
<td>17.72</td>
</tr>
<tr>
<td>$10.00$</td>
<td>16.72</td>
<td>16.82</td>
<td>18.24</td>
</tr>
<tr>
<td>$15.00$</td>
<td>17.53</td>
<td>17.37</td>
<td>18.98</td>
</tr>
<tr>
<td>$20.00$</td>
<td>18.08</td>
<td>17.91</td>
<td>19.62</td>
</tr>
<tr>
<td>$25.00$</td>
<td>18.43</td>
<td>18.39</td>
<td>20.16</td>
</tr>
<tr>
<td>$30.00$</td>
<td>18.64</td>
<td>18.73</td>
<td>20.48</td>
</tr>
</tbody>
</table>

(D3.1) Compute $\Delta t_{\text{gal}}$ values for all three supernovae, and fill them in the given blank boxes in the data tables on the BACK side of the Summary Answersheet. On a graph paper, plot the points and draw the three light curves in the rest frame (mark your graph as “D3.1”).

Solution:

Redshifts for the three supernovae are $z_1 = 0.0151$, $z_2 = 0.0314$ and $z_3 = 0.0402$.

Filling in the three tables ($\Delta t_{\text{gal}}$, third column)
Full marks of 3.5 for all correct values.
Penalty for incorrect values (3×7 independent values):

<table>
<thead>
<tr>
<th>Incorrect</th>
<th>1-3</th>
<th>4-6</th>
<th>7-9</th>
<th>10-12</th>
<th>13-15</th>
<th>16-18</th>
<th>19-21</th>
</tr>
</thead>
<tbody>
<tr>
<td>Deduction</td>
<td>0.5</td>
<td>1.0</td>
<td>1.5</td>
<td>2.0</td>
<td>2.5</td>
<td>3.0</td>
<td>3.5</td>
</tr>
</tbody>
</table>

The light curves in galaxy frame would appear as follows

Graph Number: D3.1

- Plot uses more than 50% of graph paper: 0.5
• Both axes labels ($\Delta t_{\text{gal}}$ and $m_{\text{obs}}$) present: 0.5
• Both dimensions of axes (days and mag) present: 0.5
• Ticks and values on axes (or scale written explicitly): 0.5
• Points correctly plotted:
  All points correctly plotted: 5.0
  Penalty for incorrect or missing points:

<table>
<thead>
<tr>
<th>Incorrect</th>
<th>1</th>
<th>2-4</th>
<th>5-7</th>
<th>8-10</th>
<th>11-13</th>
<th>14-16</th>
<th>17-19</th>
<th>20-22</th>
<th>23-25</th>
<th>26-30</th>
</tr>
</thead>
<tbody>
<tr>
<td>Deduction</td>
<td>0</td>
<td>0.5</td>
<td>1.0</td>
<td>1.5</td>
<td>2.0</td>
<td>2.5</td>
<td>3.0</td>
<td>3.5</td>
<td>4.0</td>
<td>5.0</td>
</tr>
</tbody>
</table>

• Smooth curve through points: 1.0 per curve

(D3.2) Take the scaling factor, $s_2$, for the supernova SN2006IS to be 1.00. Calculate the scaling factors, $s_1$ and $s_3$, for the other two supernovae SN2006TD and SN 2005LZ, respectively, by calculating $\Delta t_0$ for them.

Solution:
From the graph D3.1, SN2006IS took 22.0 d to fade by 2 magnitudes. 
That is, $\Delta t_0(\text{SN2006IS}) = 22.0$ d. 
Similarly, $\Delta t_0(\text{SN2006TD}) = 16.4$ d. 
And $\Delta t_0(\text{SN2005LZ}) = 18.8$ d. 
Acceptable range: $\pm 1.0$ days
Thus, stretching factors for these two supernovae are
\[
s_1 = \frac{22.2}{16.4} = 1.354
\]
\[
s_3 = \frac{22.2}{18.8} = 1.181
\]

(D3.3) Compute the scaled time differences, $\Delta t_s$, for all three supernovae. Write the values for $\Delta t_s$ in the same data tables on the Summary Answersheet. On another graph paper, plot the points and draw 3 light curves to verify that they now have an identical profile (mark your graph as “D3.3”).

Solution:
Filling the scaled values in the fourth column of the table ($\Delta t_s$ in table above)

Full marks of 3.5 for all correct values.
Penalty for incorrect values ($3 \times 7$ independent values):

<table>
<thead>
<tr>
<th>Incorrect</th>
<th>1-3</th>
<th>4-6</th>
<th>7-9</th>
<th>10-12</th>
<th>13-15</th>
<th>16-18</th>
<th>19-21</th>
</tr>
</thead>
<tbody>
<tr>
<td>Deduction</td>
<td>0.5</td>
<td>1.0</td>
<td>1.5</td>
<td>2.0</td>
<td>2.5</td>
<td>3.0</td>
<td>3.5</td>
</tr>
</tbody>
</table>

The scaled light curves would appear as follows, 
Graph Number: D3.3
• Plot uses more than 50% of graph paper: 0.5
• Both axes labels (\(\Delta t\) and \(m_{\text{obs}}\)) present: 0.5
• Both dimensions of axes (days and mag) present: 0.5
• Ticks and values on axes (or scale written explicitly): 0.5
• Points correctly plotted:
  - All points correctly plotted: 5.0
  - Penalty for incorrect or missing points:
    
    | Incorrect | 1 | 2-4 | 5-7 | 8-10 | 11-13 | 14-16 | 17-19 | 20-22 | 23-25 | 26-30 |
    |-----------|---|-----|-----|------|-------|-------|-------|-------|-------|-------|
    | Deduction | 0 | 0.5 | 1.0 | 1.5  | 2.0   | 2.5   | 3.0   | 3.5   | 4.0   | 5.0   |

• Smooth curve through points: 1.0 per curve
• The curves should show identical profiles.

(D3.4) Calculate the absolute magnitudes at peak brightness, \(M_{\text{peak},1}\), for SN2006TD and \(M_{\text{peak},2}\), for SN2006IS. Use these values to calculate \(a\) and \(b\).

Solution:
To get \(a\) and \(b\),

\[
M_{\text{peak},1} = m_{\text{peak},1} - \mu_1 = 15.74 - 34.27 \text{ mag} = -18.53 \text{ mag}
\]

\[
M_{\text{peak},2} = m_{\text{peak},2} - \mu_2 = 16.17 - 35.64 \text{ mag} = -19.47 \text{ mag}
\]

\[
\therefore b = \frac{s_1 - s_2}{M_{\text{peak},1} - M_{\text{peak},2}} = \frac{1.354 - 1}{-18.53 - (-19.47)} \text{ mag}^{-1} = \frac{0.354}{0.94} \text{ mag}^{-1}
\]
\( b = 0.3762 \, \text{mag}^{-1} \)

\[
a = s_2 - bM_{\text{peak,2}} = 1 - 0.3762 \times (-19.47) = 1 + 7.325
\]

\[
a = 8.325
\]

No penalty for missing \( \text{mag}^{-1} \) in \( b \).

\[ (D3.5) \] Calculate the absolute magnitude at peak brightness, \( M_{\text{peak,3}} \), and distance modulus, \( \mu_3 \), for SN2005LZ.

\[ \text{Solution:} \]

\[
s_3 = a + bM_{\text{peak,3}}
\]

\[
\therefore M_{\text{peak,3}} = \frac{s_3 - a}{b} = \frac{1.181 - 8.325}{0.3762} \, \text{mag} = \frac{-7.144}{0.3762} \, \text{mag}
\]

\[
M_{\text{peak,3}} = -18.99 \, \text{mag}
\]

Distance modulus to SN2005LZ is

\[
\mu_3 = m_{\text{peak,3}} - M_{\text{peak,3}} = 17.58 - (-18.99) \, \text{mag}
\]

\[
\mu_3 = 36.57 \, \text{mag}
\]

\[ (D3.6) \] Use the distance modulus \( \mu_3 \) to estimate the value of Hubble’s constant, \( H_0 \). Further, estimate the characteristic age of the universe, \( T_H \).

\[ \text{Solution:} \]

Distance to SN2005LZ is

\[
d_3 = 10^{\left(\frac{\mu_3}{5} + 1\right)} \, \text{pc} = 10^{\left(\frac{36.57}{5} + 1\right)} \, \text{Mpc}
\]

\[
= 10^{\left(\frac{46.57}{5}\right)} \, \text{Mpc} = 10^{2.314} \, \text{Mpc}
\]

\[
\simeq 206 \, \text{Mpc}
\]

\[
H_0 = \frac{cz_3}{d_3} = \frac{12060}{206} \, \text{km s}^{-1} \, \text{Mpc}^{-1}
\]

\[
H_0 = 58.5 \, \text{km s}^{-1} \, \text{Mpc}^{-1}
\]

\[
T_H = \frac{1}{H_0} = \frac{3.086 \times 10^{22}}{58.5 \times 10^3 \times 3.156 \times 10^7 \, \text{yr}
\]

\[
T_H = 16.7 \, \text{Gyr}
\]

Extra factor of 2/3 allowed in the value of \( T_H \).