Group Examination

(G1) A spacecraft of mass m and velocity $\vec{v}$ approaches a massive planet of mass M and orbital velocity $\vec{u}$, as measured by an inertial observer. We consider a special case, where the incoming trajectory of the spacecraft is designed in a way such that velocity vector of the planet does not change direction due to the gravitational boost given to the spacecraft. In this case, the amount of gravitational boost to the velocity the spacecraft can be roughly estimated using conservation laws by measuring asymptotic velocity of the spacecraft before and after the interaction and angle of approach of the spacecraft.

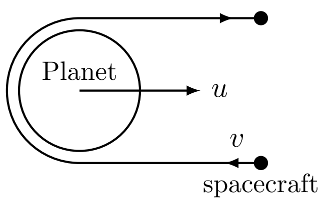


Figure 1

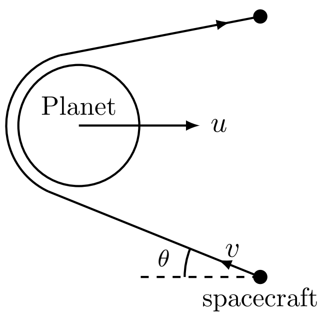


Figure 2

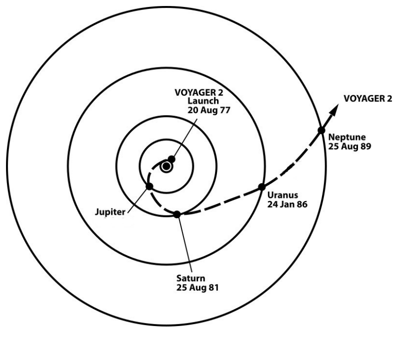


Figure 3

(G1.1) What will be the final velocity ($\vec{v_f}$) of the spacecraft, if $\vec{v}$ and $\vec{u}$ are exactly anti-parallel (see Figure 1). **3**

Solution:

Let $\vec{v_f}$ and $\vec{u_f}$ be the final velocity of the spacecraft and the planet respectively. As the planet For anti-parallel case, using conservation of linear momentum,

$$M\vec{u} + m\vec{v} = M\vec{u_f} + m\vec{v_f}$$

$$\therefore Mu - mv = Mu_f + mv_f$$

$$u_f = u - \frac{m}{M}(v_f + v)$$

Now, using conservation of energy,

$$Mu^2 + mv^2 = Mu_f^2 + mv_f^2$$

$$u^2 + \frac{m}{M}v^2 = \left(u - \frac{m}{M}(v_f + v)\right)^2 + \frac{m}{M}v_f^2$$

$$\cancel{u^2} + \cancel{\frac{m}{M}}v^2 = \cancel{u^2} + \frac{m}{M}\cdot\cancel{\frac{m}{M}}(v_f + v)^2 - 2u\cancel{\frac{m}{M}}(v_f + v) + \cancel{\frac{m}{M}}v_f^2$$

$$0 = \frac{m}{M}(v_f + v)^2 - 2u(v_f + v) + (v_f^2 - v^2)$$

$$0 = \frac{m}{M}(v_f + v)\cancel{(v_f + v)} - 2u\cancel{(v_f + v)} + \cancel{(v_f + v)}(v_f - v)$$

$$0 = \frac{m}{M}(v_f + v) - 2u + v_f - v$$

$$\therefore v_f\left(1 + \frac{m}{M}\right) = 2u + \left(1 - \frac{m}{M}\right)v$$

$$v_f = \frac{2u + \left(1 - \frac{m}{M}\right)v}{\left(1 + \frac{m}{M}\right)}$$

Alternative solution in COM frame

(G1.2) Simplify the expression for the case where $m \ll M$. **1**

Solution:
If $m \ll M$,

$$v_f \approx 2u + v$$

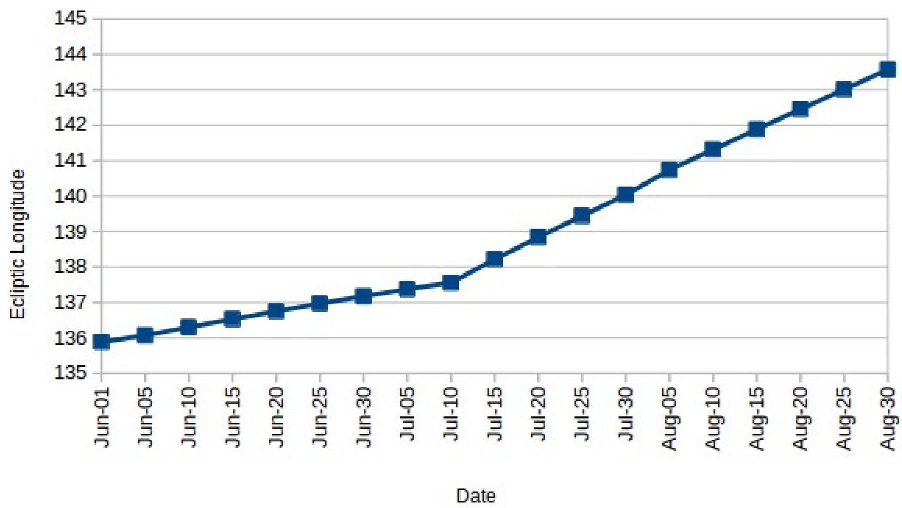
(G1.3) If angle between $\vec{v}$ and $-\vec{u}$ is θ and $m \ll M$ (see Figure 2), use results above to write expression for the magnitude of final velocity (v_f). **3**

Solution:
As velocity vector of the planet is not changing direction, there is no momentum transfer in direction perpendicular to $\vec{u}$. We will resolve $\vec{v}$ and $\vec{v_f}$ into components parallel and perpendicular to $\vec{u}$.

$$v_x = -v\cos\theta \qquad v_y = v\sin\theta$$
$$v_{f_x} = 2u + v\cos\theta \qquad v_{f_y} = v\sin\theta$$
$$v_f^2 = v_{f_x}^2 + v_{f_y}^2 = (2u + v\cos\theta)^2 + (v\sin\theta)^2$$
$$= 4u^2 + 4uv\cos\theta + v^2\cos^2\theta + v^2\sin^2\theta$$
$$= 4u^2 + 4uv\cos\theta + v^2$$
$$\therefore \boxed{v_f = \sqrt{4u^2 + v^2 + 4uv\cos\theta}}$$

(G1.4) Table on the last page gives data of Voyager-2 spacecraft for a few months in the year 1979 as it passed close to Jupiter. Assume that the observer is located at the centre of the Sun. The distance from the observer is given in AU and λ is heliocentric ecliptic longitude in degrees. Assume all objects to be in the ecliptic plane. Assume the orbit of the Earth to be circular. Plot appropriate column against the date of observation to find the date at which the spacecraft was closest to the Jupiter, and label the graph as G1.4. **8**

Solution:



From the graph, it can be inferred that the encounter with Jupiter occured on $\boxed{10^{\text{th}}\text{ July}}$ (day 191) and its distance from the Sun on that day is 5.33121 AU

(G1.5) Find the Earth-Jupiter distance, (d_{E-J}) on the day of the encounter. **4**

Solution:
The day number of Vernal Equinox is 80. Thus, ecliptic longitude of the Sun as seen from the Earth on the day of encounter will be,

$$\lambda_\odot = (191 - 80) * 360°/365.25 = 109.4045°$$

Thus, the ecliptic longitude of the Earth as seen from the Sun on the day of encounter will be,

$$\lambda_\oplus = 180° + 109.4045° = 289.4045°$$

Applying cosine rule,

$$\begin{aligned} d_{\oplus - J} &= \sqrt{d_\oplus^2 + d_J^2 - 2d_\oplus d_J \cos\Delta\lambda} \\ &= \sqrt{1^2 + 5.3312^2 - 2 \times 1 \times 5.3312 \times \cos(289.4045° - 137.5628°)} \\ &= 6.2308\,\text{AU} \end{aligned}$$

i.e. the Earth is $\boxed{6.2308\,\text{AU}}$ from Jupiter on that day.

(G1.6) On the day of the encounter, around what standard time (t_{std}) had the Jupiter transited the meridian in the sky of Bhubaneswar (20.27° N; 85.84° E; UT + 05:30)? **6**

Solution:
Thus, the angle of eastern elongation for Jupiter ($\measuredangle SEJ$) on that day would be,

$$\begin{aligned} \xi &= \sin^{-1}\left(\frac{5.3312 \times \sin(289.4045° - 137.5628°)}{6.2308}\right) \\ &= 23.8146° \end{aligned}$$

It would rise 95 minutes after the Sun rise, i.e. around 7:35am. It would transit the meridian after around 6 hours i.e. around 13:35 local time or $\boxed{\text{13:22 IST}}$.
For more precise answer, R.A. of Jupiter on the day of encounter is approximately,

$$\begin{aligned} \lambda_{J_{\text{geocentric}}} &= 109.4045° + 23.8146° = 133.2191° \\ \tan\alpha_J &= \tan\lambda_{J_{\text{geocentric}}} \cos\epsilon \\ &= \tan 133.2191° \cos 23°26' \\ \therefore \alpha_J &= 135.68° = 9^h\ 3^m \end{aligned}$$

Thus, it will culminate at that sidereal time. On that day, sidereal time at noon is 07:24 (111 days from V.E. times 4 minutes). Thus, it will culminate 1 hour 39 minutes after the local noon i.e. at 13:39 local time or at about $\boxed{\text{13:26 IST}}$.

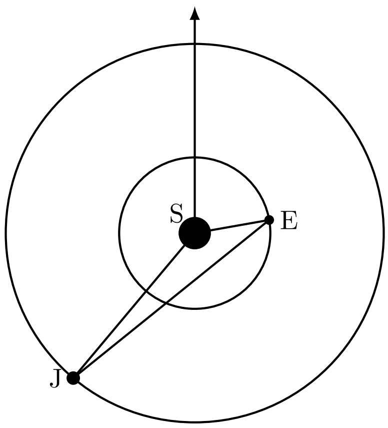


Figure 4

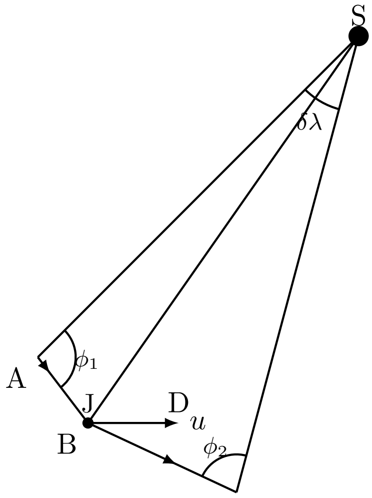


Figure 5

(G1.7) Speed of the spacecraft (in $\mathrm{km\,s^{-1}}$) as measured by the same observer on some dates before the encounter and some dates after the encounter are given below. Here day n is the date of encounter. Use these data to find the orbital speed of Jupiter (u) on the date of encounter and angle θ. **12**

date	$n-45$	$n-35$	$n-25$	$n-15$	$n-5$	n
$\mathbf{v}_{\text{tot}}$	10.1408	10.0187	9.9078	9.8389	10.2516	25.5150
date	$n+5$	$n+15$	$n+25$	$n+35$	$n+45$	
$\mathbf{v}_{\text{tot}}$	21.8636	21.7022	21.5580	21.3812	21.2365	

Solution:

In Figure 5, path of Voyager-2 is shown as A-B-C. The Sun is shown as S and the Jupiter is shown as J. From the data we note that r is increasing continuously. The same should be reflected in the diagram. For practical purpose, J and B are the same points. The direction of velocity vector of Jupiter is given by JD. In the figure,

$$\measuredangle ASB = \delta\lambda_1 \qquad \measuredangle ASB = \delta\lambda_2$$
$$\measuredangle ASC = \delta\lambda \qquad \measuredangle ABD = \theta$$
$$\measuredangle ABC = \theta_1 \qquad \measuredangle DBC = \measuredangle ABC - \measuredangle ABD = \theta_1 - \theta$$
$$\measuredangle SAB = \phi_1 \qquad \measuredangle SCB = \phi_2$$

Now the lines originating from the Sun indicate radial direction on the respective dates. Let us take speed of the spacecraft sufficiently far from the day 190, to avoid any influence of Jupiter in initial and final velocity estimation. We can choose dates 35 days on either side of July 10 i.e. June 5 and August 14.

$$\delta\lambda_1 = 137.5628^\circ - 136.0736^\circ = 1.4892^\circ$$

$$\begin{aligned} l(AB) &= \sqrt{l(SA)^2 + l(SB)^2 - 2 \times l(SA) \times l(SB) \times \cos\delta\lambda_1} \\ &= \sqrt{5.17487^2 + 5.33121^2 - 2 \times 5.17487 \times 5.33121 \times \cos 1.4892^\circ} \\ &= 0.207\,55\,\mathrm{au} \end{aligned}$$

$$\phi_1 = \sin^{-1}\left(\frac{l(SB)\sin\delta\lambda_1}{l(AB)}\right) = \sin^{-1}\left(\frac{5.33121 \times \sin 1.4892^\circ}{0.20755}\right)$$

$$
\begin{aligned}
&= \sin^{-1}(0.66755)\\
\phi_1 &= 41.8783^\circ \text{ or } \ 138.1217^\circ\\
\delta\lambda_2 &= 141.2007^\circ - 137.5628^\circ = 3.6379^\circ\\
l(BC) &= \sqrt{l(SC)^2 + l(SB)^2 - 2\times l(SC)\times l(SB)\times\cos\delta\lambda_2}\\
&= \sqrt{5.45085^2 + 5.33121^2 - 2\times 5.45085\times 5.33121\times\cos 3.6379^\circ}\\
&= 0.362\,53\,\text{au}\\
\phi_2 &= \sin^{-1}\left(\frac{l(SB)\sin\delta\lambda_2}{l(BC)}\right) = \sin^{-1}\left(\frac{5.33121\times\sin 3.6379^\circ}{0.36253}\right)\\
&= \sin^{-1}(0.66755)\\
\phi_2 &= 68.9199^\circ \text{ or } \ 111.0801^\circ
\end{aligned}
$$

from the figure, ϕ_1 should be obtuse and ϕ_2 may be acute. In $\square SABC$

$$
\begin{aligned}
\delta\lambda &= \lambda_2 - \lambda_1 = 141.2007^\circ - 136.0736^\circ\\
&= 5.1271^\circ\\
\therefore \theta_1 &= 360^\circ - \delta\lambda - \phi_1 - \phi_2\\
&= 360^\circ - 5.1271^\circ - 138.1217^\circ - 68.9199^\circ\\
\theta_1 &= 147.8313^\circ
\end{aligned}
$$

In $\triangle SBC$, we notice

$$
\begin{aligned}
\measuredangle SBC &= 180^\circ - \phi_2 - \delta\lambda_2\\
&= 180^\circ - 68.9199^\circ - 3.6379^\circ\\
&= 107.4422^\circ\\
\tan\measuredangle DBC &= \frac{v_{y_f}}{v_{x_f}} = \frac{v\sin\theta}{v\cos\theta + 2u}\\
\tan(\theta_1 - \theta) &= \frac{\sin\theta}{\cos\theta + 2\frac{u}{v}}\\
\therefore \frac{2u}{v} &= \frac{\sin\theta}{\tan(\theta_1-\theta)} - \cos\theta
\end{aligned}
$$

We use this expression to find $|\vec{u}|$.

$$
\begin{aligned}
v_f^2 &= 4u^2 + v^2 + 4uv\cos\theta\\
\frac{v_f^2}{v^2} &= \frac{4u^2}{v^2} + 1 + \frac{4u}{v}\cos\theta\\
\left(\frac{v_f}{v}\right)^2 &= \left(\frac{2u}{v}\right)^2 + 1 + 2\left(\frac{2u}{v}\right)\cos\theta\\
&= \left(\frac{\sin\theta}{\tan(\theta_1-\theta)} - \cos\theta\right)^2 + 1 + 2\left(\frac{\sin\theta}{\tan(\theta_1-\theta)} - \cos\theta\right)\cos\theta\\
&= \left(\frac{\sin\theta}{\tan(\theta_1-\theta)}\right)^2 - \cancel{\frac{2\sin\theta\cos\theta}{\tan(\theta_1-\theta)}} + \cos^2\theta + 1 + \cancel{\frac{2\sin\theta\cos\theta}{\tan(\theta_1-\theta)}} - 2\cos^2\theta\\
&= \frac{\sin^2\theta}{\tan^2(\theta_1-\theta)} + 1 - \cos^2\theta\\
&= \frac{\sin^2\theta}{\tan^2(\theta_1-\theta)} + \sin^2\theta = \sin^2\theta\left(\cot^2(\theta_1-\theta) + 1\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\sin^2\theta}{\sin^2(\theta_1-\theta)} \\
\therefore \frac{v_f}{v} &= \frac{\sin\theta}{\sin(\theta_1-\theta)} = \frac{\sin\theta}{\sin\theta_1\cos\theta - \cos\theta_1\sin\theta} \\
\therefore \frac{v}{v_f} &= \sin\theta_1\cot\theta - \cos\theta_1 \\
\tan\theta &= \frac{\sin\theta_1}{\frac{v}{v_f} + \cos\theta_1} \\
&= \frac{\sin 147.8313^\circ}{\frac{10.0187}{21.3812} + \cos 147.8313^\circ} = -1.4088 \\
\therefore \theta &= 180^\circ - 54.6328^\circ = 125.3672^\circ \\
v_f^2 &= 4u^2 + v^2 + 4uv\cos\theta \\
21.3812^2 &= 4u^2 + 10.0187^2 + 4u \times 10.0187\cos 125.3672^\circ \\
0 &= u^2 - 5.7990u - \frac{(457.1557 - 100.3743)}{4} \\
0 &= u^2 - 5.7990u - 89.1953 \\
\therefore u &= \frac{5.7990 + \sqrt{5.7990^2 + 4\times 89.1953}}{2} \\
&= 12.7789\,\mathrm{km\,s^{-1}}
\end{aligned}
$$

Jupiter's orbital velocity on the day of encounter is $\boxed{12.779\,\mathrm{km\,s^{-1}}}$ and the angle between the initial velocity of the spacecraft and Jupiter's velocity vectors is $\boxed{125^\circ 22'}$.

(G1.8) Find eccentricity, e_J, of Jupiter's orbit. **8**

Solution:
The angle between $\vec{r}$ and $\vec{u}$ on the day of encounter will be,

$$
\begin{aligned}
\psi &= \measuredangle SBC - (\theta_1 - \theta) \\
&= 107.4422^\circ - 147.8313^\circ + 125.3672^\circ \\
&= 84.9781^\circ
\end{aligned}
$$

Now we use angular momentum conservation to estimate eccentricity. If u_p and r_p represent perihelion velocity and perihelion distance of Jupiter,

$$
\begin{aligned}
r_p u_p &= a_J(1-e)\sqrt{\frac{GM_\odot}{a_J}\left(\frac{1+e}{1-e}\right)} \\
&= \sqrt{GM_\odot a_J(1-e^2)} \\
r_p u_p &= ru\sin\psi \\
\therefore 1-e^2 &= \frac{r^2u^2\sin^2\psi}{GM_\odot a_J} \\
&= \frac{5.33121^2 \times 1.496\times 10^{11} \times \left(12.7789\times 10^3\right)^2 \sin^2 84.9781^\circ}{6.6741\times 10^{-11} \times 1.9891\times 10^{30} \times 5.202\,60} \\
&= 0.99761 \\
\therefore e &= \sqrt{1 - 0.99761} = 0.0489
\end{aligned}
$$

The eccentricity of Jupiter's orbit is $\boxed{0.0489}$.

(G1.9) Find heliocentric ecliptic longitude, λ_p, of Jupiter's perihelion point. **5**

Solution:

To estimate longitude of perihelion, one should estimate true anomaly of Jupiter on that day.

$$r = \frac{a(1-e^2)}{1+e\cos\Theta}$$

$$\therefore 0.0489\cos\Theta = \frac{a(1-e^2)}{r} - 1 = \frac{5.20260 \times 0.99761}{5.33121} - 1$$

$$= -0.02646$$

$$\Theta = 122.754^\circ$$

Thus, the longitude of perihelion of Jupiter is,

$$\lambda_p = \lambda_J - \Theta$$

$$= 137.5628^\circ - 122.754^\circ$$

$$\lambda_p = 14.809^\circ$$

Month	Date	λ (^{o})	Distance (AU)
June	1	135.8870	5.1589731906
June	2	135.9339	5.1629499712
June	3	135.9806	5.1669246607
June	4	136.0272	5.1708975373
June	5	136.0736	5.1748689006
June	6	136.1200	5.1788390741
June	7	136.1662	5.1828084082
June	8	136.2122	5.1867772826
June	9	136.2582	5.1907461105
June	10	136.3040	5.1947153428
June	11	136.3496	5.1986854723
June	12	136.3951	5.2026570402
June	13	136.4405	5.2066306418
June	14	136.4857	5.2106069354
June	15	136.5307	5.2145866506
June	16	136.5756	5.2185705999
June	17	136.6202	5.2225596924
June	18	136.6647	5.2265549493
June	19	136.7090	5.2305575243
June	20	136.7532	5.2345687280
June	21	136.7970	5.2385900582
June	22	136.8407	5.2426232385
June	23	136.8841	5.2466702671
June	24	136.9273	5.2507334797
June	25	136.9702	5.2548156324
June	26	137.0127	5.2589200110
June	27	137.0550	5.2630505798
June	28	137.0969	5.2672121872
June	29	137.1384	5.2714108557
June	30	137.1795	5.2756542053
July	1	137.2200	5.2799520895
July	2	137.2600	5.2843175880
July	3	137.2993	5.2887686308
July	4	137.3378	5.2933308160
July	5	137.3754	5.2980426654
July	6	137.4118	5.3029664212
July	7	137.4467	5.3082133835
July	8	137.4798	5.3140161793
July	9	137.5116	5.3210070441
July	10	137.5628	5.3312091210
July	11	137.6898	5.3405592121
July	12	137.8266	5.3466522674
July	13	137.9599	5.3516661563
July	14	138.0903	5.3561848203
July	15	138.2186	5.3604205657
July	16	138.3453	5.3644742164

Month	Date	λ (^{o})	Distance (AU)
July	17	138.4707	5.3684017790
July	18	138.5949	5.3722377051
July	19	138.7183	5.3760047603
July	20	138.8409	5.3797188059
July	21	138.9628	5.3833913528
July	22	139.0841	5.3870310297
July	23	139.2048	5.390644477
July	24	139.3250	5.3942369174
July	25	139.4448	5.3978125344
July	26	139.5641	5.4013747321
July	27	139.6831	5.4049263181
July	28	139.8016	5.4084696349
July	29	139.9198	5.4120066575
July	30	140.0377	5.4155390662
July	31	140.1553	5.4190683021
August	1	140.2725	5.4225956100
August	2	140.3895	5.4261220723
August	3	140.5062	5.4296486357
August	4	140.6225	5.4331761326
August	5	140.7387	5.4367052982
August	6	140.8546	5.4402367851
August	7	140.9702	5.4437711745
August	8	141.0856	5.4473089863
August	9	141.2007	5.4508506867
August	10	141.3157	5.4543966955
August	11	141.4303	5.4579473912
August	12	141.5448	5.4615031166
August	13	141.6591	5.4650641822
August	14	141.7731	5.4686308707
August	15	141.8869	5.4722034391
August	16	142.0006	5.4757821220
August	17	142.1140	5.4793671340
August	18	142.2272	5.4829586711
August	19	142.3402	5.4865569133
August	20	142.4530	5.4901620256
August	21	142.5657	5.4937741595
August	22	142.6781	5.4973934544
August	23	142.7904	5.5010200385
August	24	142.9024	5.5046540300
August	25	143.0143	5.5082955377
August	26	143.1260	5.5119446617
August	27	143.2375	5.5156014948
August	28	143.3488	5.5192661222
August	29	143.4599	5.5229386226
August	30	143.5709	5.5266190687
August	31	143.6817	5.5303075275